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Moderate deviations for martingales and mixing random processes[1]

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Abstract

We obtain a moderately large deviation theorem for martingales. Then this result is applied to prove that the empirical measures of a stationary ϕ-mixing sequence of random variables satisfy moderately large deviation principle when $\sum_{n=1}^{+\infty} \phi(n) < +\infty$. Another application shows that the empirical measures of a Markov process obey uniformly moderately large deviation principle under Doeblin recurrence.

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1. Introduction and main results

In recent years, a lot of works have devoted to the study of the moderate deviations (cf. Chen, 1991; de Acosta, 1992; Ledoux, 1992; Mogulskii, 1984; Wu, 1992; Gao, 1992 etc.). However, few papers deal with martingales and mixing random processes until now. Wu (1992) introduced the notion of C^2-regularity and proved that the C^2-regularity implies moderate deviations. But, it is difficult to show that a martingale or a mixing random process satisfies the C^2-regularity condition. In this paper, we discuss the moderate deviations for martingales and mixing random processes by a different approach.

Let $\{X_n, n \in \mathbb{Z}\}$ be a sequence of random variables with values in a Polish space (S, ρ) on some probability space $(\Omega, \mathscr{F}, P)$. Denote by σ-fields $\mathscr{F}_t^s = \sigma(X_u, s \leqslant u \leqslant t)$, $\mathscr{F}_t = \sigma(X_u, u \leqslant t)$ and $\mathscr{F}^s = \sigma(X_u, u \geqslant s)$. Set

$$\phi(n) = \sup(|P(B|A) - P(B)|;\ A \in \mathscr{F}_k \text{ with } P(A) > 0,\ B \in \mathscr{F}^{k+n},\ k \in \mathbb{Z}).$$

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The sequence $\{X_n, n \in \mathbb{Z}\}$ is said to be ϕ-mixing if $\phi(n) \to 0$ as $n \to +\infty$. Throughout this paper, we let $a(t)$, $t > 0$ be a positive function satisfying

$$\lim_{t\to+\infty} \frac{t}{a(t)} = +\infty \quad \text{and} \quad \lim_{t\to+\infty} \frac{a(t)}{\sqrt{t}} = +\infty.$$

Theorem 1.1. *Let A be a non-empty set and let for each $\alpha \in A$, $\{X_{\alpha,n}, \mathscr{F}_{\alpha,n}, P_\alpha; n \in \mathbb{Z}\}$ be martingale difference sequence with values in $\mathbb{R}^d$. Suppose that*

(A.1) *there exists a $\delta > 0$ such that*

$$\sup_{\alpha\in A}\sup_{m\geqslant 0} \| E^{P_\alpha}(\exp(\delta|X_{\alpha,m+1}|)|\mathscr{F}_{\alpha,m})\|_{L^\infty(P_\alpha)} < +\infty. \tag{1}$$

(A.2) *there exists a non-negative definite matrix $\sigma =: (\sigma_{ij})_{d\times d}$ such that for any $x \in \mathbb{R}^d$,*

$$\liminf_{n\to+\infty,\frac{n}{m}\to 0}\ \sup_{\alpha\in A}\sup_{j\geqslant 0} \left\| \frac{1}{m} E^{P_\alpha}\left(\sum_{i=1}^{m} \langle X_{\alpha,n+j+i}, x\rangle^2 | \mathscr{F}_{\alpha,j}\right) - \langle x, \sigma x\rangle \right\|_{L^\infty(P_\alpha)} = 0. \tag{2}$$

Then for any $x \in \mathbb{R}^d$,

$$\lim_{n\to+\infty}\sup_{\alpha\in A}\left| \frac{n}{a^2(n)} \log E^{P_\alpha} \exp\left(\frac{a(n)}{n}\sum_{i=1}^{n}\langle X_{\alpha,i}, x\rangle\right) - \frac{1}{2}\langle x, \sigma x\rangle \right| = 0; \tag{3}$$

and for any Borel set $B \subseteq \mathbb{R}^d$,

$$\limsup_{n\to+\infty} \frac{n}{a^2(n)} \log \sup_{\alpha\in A} P_\alpha\left(\frac{1}{a(n)}\sum_{i=1}^{n} X_{\alpha,i} \in B\right) \leqslant -\inf_{x\in\bar{B}} J(x), \tag{4}$$

$$\liminf_{n\to+\infty} \frac{n}{a^2(n)} \log \inf_{\alpha\in A} P_\alpha\left(\frac{1}{a(n)}\sum_{i=1}^{n} X_{\alpha,i} \in B\right) \geqslant -\inf_{x\in\mathring{B}} J(x), \tag{5}$$

where

$$J(x) = \sup_{y\in\mathbb{R}^d}(\langle x, y\rangle - \langle y, \sigma y\rangle). \tag{6}$$

Corollary 1.1. *Let $\{X_n, \mathscr{F}_n, P\}$ be a ϕ-mixing, stationary, bounded martingale difference sequence taking its values in $\mathbb{R}^d$. Then for any $x \in \mathbb{R}^d$,*

$$\lim_{n\to+\infty} \frac{n}{a^2(n)} \log E \exp\left(\frac{a(n)}{n}\sum_{i=1}^{n}\langle X_i, x\rangle\right) = \frac{1}{2} E\langle X_0, x\rangle^2; \tag{7}$$

and for any Borel set $B \subseteq \mathbb{R}^d$

$$\limsup_{n\to+\infty} \frac{n}{a^2(n)} \log P\left(\frac{1}{a(n)}\sum_{i=1}^{n} X_i \in B\right) \leqslant -\inf_{x\in\bar{B}} J(x), \tag{8}$$

$$\liminf_{n\to+\infty} \frac{n}{a^2(n)} \log P\left(\frac{1}{a(n)}\sum_{i=1}^{n} X_i \in B\right) \geqslant -\inf_{x\in\mathring{B}} J(x), \tag{9}$$

where

$$J(x) = \sup_{y \in \mathbb{R}^d} \left(\langle x, y \rangle - \frac{1}{2} E(\langle X_0, x \rangle^2) \right). \tag{10}$$

Let us denote by $M(S)$ the space of all real measures of finite variation on $(S, \mathscr{B}(S))$, and denote by $\mathscr{B}_b(S)$ the space of bounded measurable functions on $(S, \mathscr{B}(S))$. The τ-topology on $M(S)$ is induced by the mappings $M(S) \ni \nu \mapsto \langle f, \nu \rangle \equiv \int f \, d\nu$, $f \in \mathscr{B}_b(S)$. We also endow $M(S)$ with σ-fields D_s generated by these mappings.

Theorem 1.2. *Let $\{X_n, \mathscr{F}_n, P\}$ be a ϕ-mixing stationary sequence and let μ denote the distribution of X_1. If*

$$\text{(A.3)} \qquad \sum_{n=1}^{+\infty} \phi(n) < +\infty,$$

then for any $f \in \mathscr{B}_b(S)$,

$$\Lambda(f) \equiv \lim_{n \to +\infty} \frac{n}{a^2(n)} \log E \exp\left(\frac{a(n)}{n} \sum_{i=1}^{n} (f(X_i) - Ef(X_i)) \right) \tag{11}$$

$$= \frac{1}{2} \lim_{n \to +\infty} \frac{1}{n} E\left[\sum_{n=1}^{n} (f(X_i) - Ef(X_i)) \right]^2, \tag{12}$$

exists and for any $B \in D_s$,

$$\limsup_{n \to +\infty} \frac{n}{a^2(n)} \log P\left(\frac{1}{a(n)} \sum_{i=1}^{n} \delta_{X_i} - n\mu \in B \right) \leqslant - \inf_{v \in \bar{B}} \Lambda^*(v), \tag{13}$$

$$\liminf_{n \to +\infty} \frac{n}{a^2(n)} \log P\left(\frac{1}{a(n)} \sum_{i=1}^{n} \delta_{X_i} - n\mu \in B \right) \geqslant - \inf_{v \in \mathring{B}} \Lambda^*(v), \tag{14}$$

where

$$\Lambda^*(v) = \sup_{f \in \mathscr{B}_b(S)} (\langle f, v \rangle - \Lambda(f)). \tag{15}$$

Theorem 1.3. *Let $X \equiv (X_n, \mathscr{F}_n, (P_x)_{x \in S})$ be a Markov chain taking its values in a Polish space (S, ρ) with Markov kernel $\pi(x, dy)$. If X is Harris recurrent and satisfies Doeblin's condition (cf. Revuz, 1984), then for any $f \in \mathscr{B}_b(S)$,*

$$\lim_{n \to +\infty} \sup_{x \in S} \left| \frac{n}{a^2(n)} \log E^{P_x} \exp\left(\frac{a(n)}{n} \sum_{i=1}^{n} (f(X_i) - \langle f, \mu \rangle) \right) - \Lambda(f) \right| = 0; \tag{16}$$

and for any $B \in D_s$,

$$\limsup_{n \to +\infty} \frac{n}{a^2(n)} \log \sup_{x \in S} P_x\left(\frac{1}{a(n)} \sum_{i=1}^{n} \delta_{X_i} - n\mu \in B \right) \leqslant - \inf_{v \in \bar{B}} \Lambda^*(v), \tag{17}$$

$$\liminf_{n \to +\infty} \frac{n}{a^2(n)} \log \inf_{x \in S} P_x\left(\frac{1}{a(n)} \sum_{i=1}^{n} \delta_{X_i} - n\mu \in B \right) \geqslant - \inf_{v \in \mathring{B}} \Lambda^*(v), \tag{18}$$

where μ denote the invariant probability measure of X and

$$\Lambda(f) = \frac{1}{2}\langle ((I-\pi)^{-1}(f - \langle f, \mu\rangle))^2 - (\pi(I-\pi)^{-1}(f - \langle f, \mu\rangle))^2, \mu\rangle, \tag{19}$$

$$(I-\pi)f(x) \equiv f(x) - \pi f(x), \tag{20}$$

$$\Lambda^*(v) = \sup_{f \in \mathscr{B}_b(S)} (\langle f, v\rangle - \Lambda(f)). \tag{21}$$

The proof of Theorem 1.1 and Corollary 1.1 will be given in Section 2. We will prove Theorems 1.2 and 1.3 in Sections 3 and 4, respectively.

Remarks. (a) Theorem 1.3 improves the results in Mogulskii (1984) and Gao (1992). Gao (1995) showed that the Doeblin recurrence is also a necessary condition for the uniform moderate deviation principle of the empirical measures of the Markov chain. For the aperiodic case, Wu (1992) discussed moderate deviations for Markov processes with exponential convergence by analytic perturbation theory of operators. The proof here is different from his.

(b) Theorems 1.1 can easily be extended to random processes with continuous time parameters.

Theorem 1.1′. *Let for each $\alpha \in A$, $\{Y_{\alpha,t}, \mathscr{F}_{\alpha,t}, P_\alpha\}$, $(t \in R_+)$ be a martingale with values in $\mathbb{R}^d$. Suppose that $Y_{\alpha,0} = 0$ and*

(A.1)′ *there exists a $\delta > 0$ such that*

$$\sup_{\alpha\in A}\sup_{s\geqslant 0}\left\| E^{P_\alpha}\left(\exp\left(\delta \sup_{s\leqslant u\leqslant s+1} |Y_{\alpha,u} - Y_{\alpha,s}|\right)\middle|\mathscr{F}_{\alpha,s}\right)\right\|_{L^{+\infty}(P_\alpha)} < +\infty,$$

(A.2)′ *there exists a non-negative definite matrix $\sigma = (\sigma_{ij})_{d\times d}$ such that for any $x \in \mathbb{R}^d$,*

$$\liminf_{s\to+\infty,\frac{s}{t}\to 0}\ \sup_{\alpha\in A, u\geqslant 0}\left\|\frac{1}{t}E^{P_\alpha}(\langle Y_{\alpha,s+t+u} - Y_{\alpha,s+u}, x\rangle^2 \mid \mathscr{F}_{\alpha,u}) - \langle x, \sigma x\rangle\right\|_{L^{+\infty}(P_\alpha)} = 0.$$

Then for any $x \in \mathbb{R}^d$

$$\lim_{t\to+\infty}\sup_{\alpha\in A}\left|\frac{t}{a^2(t)}\log E^{P_\alpha}\left(\exp\left(\frac{a(t)}{t}\langle Y_{\alpha,t}, x\rangle\right)\right) - \frac{1}{2}\langle x, \sigma x\rangle\right| = 0;$$

and for any Borel set $B \subseteq \mathbb{R}^d$,

$$\limsup_{t\to+\infty}\frac{t}{a^2(t)}\log\sup_{\alpha\in A}P_\alpha\left(\frac{1}{a(t)}Y_{\alpha,t} \in B\right) \leqslant -\inf_{x\in\bar{B}} J(x),$$

$$\liminf_{t\to+\infty}\frac{t}{a^2(t)}\log\inf_{\alpha\in A}P_\alpha\left(\frac{1}{a(t)}Y_{\alpha,t} \in B\right) \geqslant -\inf_{x\in\mathring{B}} J(x),$$

where

$$J(x) = \sup_{y\in\mathbb{R}^d}\left(\langle x, y\rangle - \frac{1}{2}\langle y, \sigma y\rangle\right).$$

2. Proofs of Theorem 1.1 and Corollary 1.1

Lemma 2.1. *Let for each $\alpha \in A$, $\{X_{\alpha,n}, \mathscr{F}_{\alpha,n}, P_\alpha\}$ be a sequence of random variables with values in $\mathbb{R}^d$. If assumption* (A.1) *holds, then for any integer $M \geqslant 1$; there exists $0 < C(\delta, M) < +\infty)$ such that for all $\alpha \in A$, $n \geqslant 0$ and $m \geqslant 0$,*

$$\left\| E^{P_\alpha}\left(\left(\sum_{i=n+1}^{n+M} |X_{\alpha,m+i}|\right)^k \exp\left(\frac{\delta}{2}\sum_{i=n+1}^{n+M} |X_{\alpha,m+i}|\right) \Bigg| \mathscr{F}_{\alpha,m}\right)\right\|_{L^\infty(P_\alpha)} \leqslant C(\delta, M)$$

for $k = 0, 1, 2, 3$.

Proof. Using (A.1) and the inequality $1 + x^2 + x^3 \leqslant 9\mathrm{e}^x$ $(x \geqslant 0)$, we can easily get this lemma. □

Proof of Theorem 1.1. In order to prove Theorem 1.1, by Theorem 3.4 in Dawson and Gärtner (1987), it is enough to prove (3). For fixed $x \in \mathbb{R}^d$, write $Y_i^\alpha = \langle X_{\alpha,i}, x\rangle$ and

$$\sigma^2_{M,N,j,\alpha} = \left\| E^{P_\alpha}\left(\sum_{i=N+1}^{N+M} (Y^\alpha_{j+1})^2 \,|\, \mathscr{F}_{\alpha,j}\right)\right\|_{L^{+\infty}(P_\alpha)}.$$

For fixed $N \geqslant 1$ and $M \geqslant 1$, set

$$k \equiv k(n) = [n/(N+M)] \text{ and } r \equiv r(n) = n - k(N+M), \qquad n \geqslant 1.$$

Then for $n \geqslant 1$ large enough, by Hölder inequality, we have that

$$\begin{aligned}
&E^{P_\alpha}\left(\exp\left(\frac{a(n)}{n}\sum_{i=1}^{n} Y_i^\alpha\right)\right)\\
&\leqslant C(\delta, M+N) E^{P_\alpha}\left(\exp\left(\frac{a(n)}{n}\sum_{j=0}^{k-1}\sum_{i=1}^{M} Y^\alpha_{j(N+M)+i} + \frac{a(n)}{n}\sum_{j=0}^{k-1}\sum_{i=1}^{N} Y^\alpha_{j(N+M)+M+i}\right)\right)\\
&\leqslant C(\delta, M+N)\left\|\exp\left(\frac{a(n)}{n}\sum_{j=0}^{k-1}\sum_{i=1}^{M} Y^\alpha_{j(N+M)+i}\right)\right\|_{L^p(P_\alpha)}\\
&\quad\times\left\|\exp\left(\frac{a(n)}{n}\sum_{j=0}^{k-1}\sum_{i=1}^{N} Y^\alpha_{j(N+M)+M+i}\right)\right\|_{L^q(P_\alpha)},
\end{aligned}$$

where $1/p + 1/q = 1$ $(p > 1, q > 1)$. For $n \geqslant 1$ satisfying $|x|pa(n)/n < \delta/2$ and $|x|qa(n)/n < \delta/2$, we have that

$$\begin{aligned}
&\left\|\exp\left(\frac{a(n)}{n}\sum_{j=0}^{k-1}\sum_{i=1}^{M} Y^\alpha_{j(N+M)+i}\right)\right\|^p_{L^p(P_\alpha)}\\
&\leqslant \left\|\exp\left(\frac{a(n)}{n}\sum_{j=0}^{k-2}\sum_{i=1}^{M} Y^\alpha_{j(N+M)+i}\right)\right\|^p_{L^p(P_\alpha)}\left(1 + \frac{1}{2}\left(\frac{pa(n)}{n}\right)^2 \sup_{\alpha\in A, j\geqslant 0} \sigma^2_{M,N,j,\alpha}\right.\\
&\quad\left. + \frac{C(\delta, M)}{6}\left(\frac{pa(n)}{n}\right)^3\right)\\
&\leqslant \left(1 + \frac{1}{2}\left(\frac{pa(n)}{n}\right)^2 \sup_{\alpha\in A, j\geqslant 0} \sigma^2_{M,N,j,\alpha} + \frac{C(\delta, M)}{6}\left(\frac{pa(n)}{n}\right)^3\right)^k,
\end{aligned}$$

and

$$\left\| \exp\left(\frac{a(n)}{n}\sum_{j=0}^{k-1}\sum_{i=1}^{N} Y^{\alpha}_{j(N+M)+M+i}\right)\right\|^{q}_{L^q(P_\alpha)} \leqslant \left(1+\frac{C(\delta,1)}{2}\left(\frac{qa(n)}{n}\right)^2\right)^{Nk}.$$

Hence,

$$\limsup_{n\to+\infty}\frac{n}{a^2(n)}\log\sup_{\alpha\in A} E^{P_\alpha}\left(\exp\left(\frac{a(n)}{n}\sum_{i=1}^{n} Y^{\alpha}_i\right)\right)$$
$$\leqslant \frac{p}{2(N+M)}\sup_{\alpha\in A, j\geqslant 0}\sigma^2_{M,N,j,\alpha}+\frac{NqC(\delta,1)}{2(N+M)},$$

this and (A.2) imply

$$\limsup_{n\to+\infty}\frac{n}{a^2(n)}\log\sup_{\alpha\in A} E^{P_\alpha}\left(\exp\left(\frac{a(n)}{n}\sum_{i=1}^{n} Y^{\alpha}_i\right)\right)\leqslant \frac{p}{2}\langle x,\sigma x\rangle$$

for all $p>1$. Therefore,

$$\limsup_{n\to+\infty}\frac{n}{a^2(n)}\log\sup_{\alpha\in A} E^{P_\alpha}\left(\exp\left(\frac{a(n)}{n}\sum_{i=1}^{n} Y^{\alpha}_i\right)\right)\leqslant \frac{1}{2}\langle x,\sigma x\rangle. \tag{22}$$

On the other hand, for any $p>1$ and $q>1$ satisfying $1/p+1/q=1$, any $n\geqslant 1$ satisfying $|x|qa(n)/(np)<\delta/2$, we have that

$$E^{P_\alpha}\left(\exp\left(\frac{a(n)}{np}\sum_{j=0}^{k-1}\sum_{i=1}^{M} Y^{\alpha}_{j(N+M)+i}\right)\right)$$
$$\leqslant C(\delta,N+M)^{1/q}\left(E^{P_\alpha}\left(\exp\left(\frac{a(n)}{n}\sum_{i=1}^{n} Y^{\alpha}_i\right)\right)\right)^{1/p}$$
$$\times\left(E^{P_\alpha}\left(\exp\left(-\frac{qa(n)}{np}\sum_{j=0}^{k-1}\sum_{i=1}^{N} Y^{\alpha}_{j(N+M)+M+i}\right)\right)\right)^{1/q},$$

$$E^{P_\alpha}\left(\exp\left(\frac{a(n)}{np}\sum_{j=0}^{k-1}\sum_{i=1}^{M} Y^{\alpha}_{j(N+M)+i}\right)\right)$$
$$\geqslant\left(1+\frac{1}{2}\left(\frac{a(n)}{np}\right)^2\inf_{\alpha\in A, j\geqslant 0}\sigma^2_{M,N,j,\alpha}\frac{C(\delta,M)}{6}\left(\frac{a(n)}{pn}\right)^3\right)^{k},$$

and

$$E^{P_\alpha}\left(\exp\left(-\frac{a(n)q}{np}\sum_{j=0}^{k-1}\sum_{i=1}^{N} Y^{\alpha}_{j(N+M)+M+i}\right)\right)\leqslant\left(1+\frac{C(\delta,1)}{2}\left(\frac{qa(n)}{np}\right)^2\right)^{Nk}.$$

Hence,

$$\liminf_{n\to+\infty}\frac{n}{a^2(n)}\log\inf_{\alpha\in A} E^{P_\alpha}\left(\exp\left(\frac{a(n)}{n}\sum_{i=1}^{n} Y^{\alpha}_i\right)\right)$$
$$\geqslant\frac{1}{2p(N+M)}\inf_{\alpha\in A, j\geqslant 0}\sigma^2_{M,N,j,\alpha}-\frac{NqC(\delta,1)}{2(N+M)p},$$

this and (A.2) imply

$$\liminf_{n\to+\infty}\frac{n}{a^2(n)}\log\inf_{\alpha\in A}E^{P_\alpha}\left(\exp\left(\frac{a(n)}{n}\sum_{i=1}^{n}Y_i^\alpha\right)\right)\geqslant\frac{1}{2p}\langle x,\sigma x\rangle$$

for all $p>1$. Hence,

$$\liminf_{n\to+\infty}\frac{n}{a^2(n)}\log\inf_{\alpha\in A}E^{P_\alpha}\left(\exp\left(\frac{a(n)}{n}\sum_{i=1}^{n}Y_i^\alpha\right)\right)\geqslant\frac{1}{2}\langle x,\sigma x\rangle. \tag{23}$$

From (22) and (23), we obtain (3), and so Theorem 1.1 holds. □

Proof of Corollary 1.1. Since $\{X_n,\mathscr{F}_n,P\}$ is bounded, stationary and ϕ-mixing, we can easily get that

$$\sup_m\|E(\exp(\delta|X_{m+1}|)\,|\,\mathscr{F}_m)\|_{L^{+\infty}(P)}<+\infty\quad\text{for all }\delta>0,$$

and

$$\begin{aligned}&\left|\frac{1}{m}E\left(\sum_{i=1}^{m}\langle X_{n+j+i},x\rangle^2\,|\,\mathscr{F}_j\right)-E(\langle X_0,x\rangle^2)\right|\\&\quad\leqslant\left|\frac{1}{m}E\left(\sum_{i=1}^{m}\langle X_{n+j+i},x\rangle^2\right)-E(\langle X_0,x\rangle^2)\right|+2C^2|x|^2\phi(n)\\&\quad=2C^2|x|^2\phi(n)\end{aligned}$$

for all $n\geqslant1$, $m\geqslant1$, $j\geqslant0$ and $x\in\mathbb{R}^d$. Hence, by Theorem 1.1, we get Corollary 1.1. □

3. Proof of Theorem 1.2

Without loss of generality, we assume that $\Omega=S^{\mathbb{Z}}$, $\mathscr{F}=:\sigma(\omega(n),\ n\in\mathbb{Z})$ and $X_n(\omega)=\omega(n), n\in\mathbb{Z}, \omega=(\omega(j), j\in\mathbb{Z})\in\Omega$. Let θ denote the backward shift defined by

$$(\theta\omega)(n)=\omega(n+1),\quad n\in\mathbb{Z},$$

and let P be a stationary probability measure on $(\Omega,\mathscr{F})$. For every random variable X on $(\Omega,\mathscr{F})$, we define as usual the transformation U:

$$(UX)(\omega)=X(\theta\omega).$$

As in Section 1, we define

$$\phi(n)=\sup(|P(B\,|\,A)-P(B)|;\ A\in\mathscr{F}_0\text{ with }P(A)>0,\ B\in\mathscr{F}^n).$$

Throughout this section, we assume that (A.3) holds, i.e.,

$$\sum_{n=1}^{+\infty}\phi(n)<+\infty.$$

For each $k \geqslant 1$, let $p_k(\omega, \mathrm{d}x)$ denote the regular conditional probability distribution of random variable X_k given the σ-field $\mathscr{F}_{-1}$, i.e.,

(a) for each $w \in \Omega$, $p_k(w, \mathrm{d}x)$ is a probability measure on $\mathscr{B}(S)$;

(b) for any $A \in \mathscr{B}(S)$, the map $\omega \mapsto p_k(w, A)$ is $\mathscr{F}_{-1}$-measurable;

(c) for any $A \in \mathscr{B}(S)$, $B \in \mathscr{F}_{-1}$

$$\int_B p_k(w, A) P(\mathrm{d}w) = P(B \cap \{X_k \in A\}).$$

and let $q_k(\omega, \mathrm{d}x)$ denote the regular conditional probability distribution of random variable X_k given the σ-field F_0.

Lemma 3.1. *For any* $A \in \mathscr{B}(S)$,

$$\sum_{k=0}^{+\infty} \| p_k(\cdot, A) - q_k(\cdot, A) \|_{L^{+\infty}(P)} \leqslant 4 \sum_{k=0}^{+\infty} \phi(k).$$

Proof.

$$\begin{aligned} &\| p_k(\cdot, A) - q_k(\cdot, A) \|_{L^\infty(P)} \\ &\quad \leqslant \| P(X_k \in A \mid \mathscr{F}_{-1}) - P(X_k \in A) \|_{L^\infty(P)} + \| P(X_k \in A \mid \mathscr{F}_0) - P(X_k \in A) \|_{L^\infty(P)} \\ &\quad \leqslant 2\phi(k+1) + 2\phi(k) \leqslant 4\phi(k). \end{aligned}$$

The lemma is proved. □

Define

$$r(w, \mathrm{d}x) = \sum_{k=0}^{+\infty} [p_k(w, \mathrm{d}x) - q_k(w, \mathrm{d}x)],$$

and for any $f \in \mathscr{B}_b(S)$,

$$Rf(w) = \int_S f(x) r(w, \mathrm{d}x), \tag{24}$$

$$\Lambda(f) = \frac{1}{2} \int_\Omega [Rf(w)]^2 P(\mathrm{d}w), \tag{25}$$

where $\mathscr{B}_b(S)$ denotes the space of bounded $\mathscr{B}(S)$-measurable functions on $(S, \mathscr{B}(S))$. For each $v \in \mathscr{B}_b^*(S) \equiv$ the algebraic dual space of $\mathscr{B}_b(S)$, define

$$I(v) = \sup_{f \in \mathscr{B}_b(S)} (\langle f, v \rangle - \Lambda(f)) \tag{26}$$

where $\langle \cdot, \cdot \rangle$ is the dual form on $\mathscr{B}_b(S) \times \mathscr{B}_b^*(S)$.

Lemma 3.2. (a) *For any sequence* $\{f_n\} \subset \mathscr{B}_b(S)$ *satisfying* $\sup_n \sup_{x \in S} |f_n(x)| < +\infty$ *and* $\lim_{n \to +\infty} f_n(x) = 0$ $(\forall x \in S)$,

$$\lim_{n \to +\infty} \Lambda(f_n) = 0.$$

(b) $\{\nu; I(\nu) < +\infty\} \subset M(S)$, *where* $M(S)$ *denotes the space of all real measures with finite variation.*

Proof. We prove first (a). By Lemma 3.1, it is easy to prove

$$\sup_n \|Rf_n\|_{L^{+\infty}(P)} < +\infty \text{ and } \lim_{n\to+\infty} Rf_n = 0, \qquad P - a.e.$$

Hence, $\lim_{n\to+\infty} \Lambda(f_n) = 0$. Now, we prove (b). Given ν with $I(\nu) < +\infty$, in order to prove $\nu \in M(S)$, it is enough to show that

$$\lim_{n\to+\infty} \langle f_n, \nu\rangle = 0$$

for any sequence $\{f_n\} \in \mathscr{B}_b(S)$ satisfying $\sup_n \sup_x |f_n(x)| < +\infty$ and $\lim_{n\to+\infty} f_n(x) = 0$ ($\forall x \in S$). In fact, for any $c > 0$

$$|\langle f_n, \nu\rangle| \leqslant (I(\nu) + \Lambda(c f_n))/c.$$

Hence, by (a) we have

$$\limsup_{n\to+\infty} |\langle f_n, \nu\rangle| \leqslant I(\nu)/c, \quad c > 0,$$

this implies that $\lim_{n\to+\infty} \langle f_n, \nu\rangle = 0$. $\square$

Lemma 3.3. *Let* $f \in \mathscr{B}_b(S)$. *Define*

$$Z_1 = \sum_{k=0}^{+\infty} [U^k E^P(f(X_1) | \mathscr{F}_{-k}) - E^P f(X_1)],$$

and

$$Y_1 = \sum_{k=0}^{+\infty} U^k [E^P(f(X_1) | \mathscr{F}_{-k+1}) - E^P(f(X_1) | \mathscr{F}_{-k})].$$

Then $\|Z_1\|_{L^{+\infty}(P)} < +\infty$, $f(X_1) - E^P f(X_1) = Y_1 - UZ_1 + Z_1$, *and* $\{U^n Y_1, \mathscr{F}_n, P; n \geqslant 0\}$ *is a bounded, ϕ-mixing, stationary martingale difference sequence.*

Proof. We prove only $\|Z_1\|_{L^{+\infty}(P)} < +\infty$. Since

$$U^k E^P(f(X_1) | \mathscr{F}_{-k}) = E^P(f(X_{k+1}) | \mathscr{F}_0)$$

and

$$\|E^P(f(X_{k+1}) | \mathscr{F}_0) - E^P(f(X_{k+1}))\|_{L^{+\infty}(P)} \leqslant 2\phi(k+1) \sup_x |f(x)|,$$

we have

$$\|Z_1\|_{L^{+\infty}(P)} \leqslant 2 \sup_x |f(x)| \sum_{k=1}^{+\infty} \phi(k) < +\infty. \qquad \square$$

Lemma 3.4. *For any* $f \in \mathscr{B}_b(S)$,

$$\Lambda(f) = \lim_{n\to+\infty} \frac{n}{a^2(n)} \log E^P\left(\exp\left(\frac{a(n)}{n}\sum_{i=1}^n (f(X_i) - E^P(f(X_i))\right)\right)$$
$$= \frac{1}{2}\lim_{n\to+\infty}\frac{1}{n}E^P\left(\sum_{n=1}^n (f(X_i) - E^P f(X_i))\right)^2$$

Proof. Let Y_1 and Z_1 be defined as same as in Lemma 3.3. Then

$$\frac{n}{a^2(n)}\left|\log E^P\left(\exp\left(\frac{a(n)}{n}\sum_{i=1}^n (f(X_i) - E^P f(X_i))\right)\right)\right.$$
$$\left. - \log E^P\left(\exp\left(\frac{a(n)}{n}\sum_{i=1}^n U^i Y_1\right)\right)\right|$$
$$\leqslant \frac{2\|Z_1\|_{L^{+\infty}(P)}}{a(n)}$$

and

$$\left|\frac{1}{n}E^P\left(\sum_{i=1}^n f(X_i)\right)^2 - EY_1^2\right| \leqslant \frac{4\|Z_1\|^2_{L_\infty(P)}}{n} + \frac{4\|Z_1\|_{L_\infty(P)}E|\sum_{i=1}^n U^i Y_1|}{n}.$$

Hence

$$\lim_{n\to+\infty}\frac{n}{a^2(n)}\log E^P\left(\exp\left(\frac{a(n)}{n}\sum_{i=1}^n (f(X_i) - E^P(f(X_i))\right)\right)$$
$$= \frac{1}{2}\lim_{n\to+\infty}\frac{1}{n}E^P\left(\sum_{n=1}^n f(X_i) - E^P(f(X_i))\right)^2$$
$$= \frac{1}{2}EY_1^2$$
$$= \Lambda(f). \qquad \square$$

Proof of Theorem 1.2. By Theorem 3.4 in Dawson and Gärtner (1987), Lemmas 3.2 and 3.4 imply Theorem 1.2. $\square$

4. Proof of Theorem 1.3

Throughout this section, let $X \equiv (X_n, \mathscr{F}_n, (P_x)_{x\in S})$ be a Markov chain taking its values in a Polish space (S,ρ) with Markov kernel $\pi(x, \mathrm{d}y)$ and let X be Harris recurrent and satisfy Doeblin condition. It is well-known under these assumptions that (cf. Revuz, 1984)

(a) there is only invariant probability measure μ such that

$$\lim_{n\to+\infty}\sup_{x\in S}\sup_{\|f\|\leqslant 1}\left|\frac{1}{n}\sum_{k=1}^n \pi^k f(x) - \langle f,\mu\rangle\right| = 0,$$

where $\| f \| = \sup_{x \in S} | f(x)|$, $\pi^0 f(x) = f(x)$, $\pi f(x) = \int_S f(y)\pi(x, \mathrm{d}y)$ and $\pi^{k+1} f(x) = \pi(\pi^k f)(x)$ $(\forall k \geqslant 0)$;

(b) the operator $I - \pi$: $(I - \pi) f(x) = f(x) - \pi f(x)$, $f \in \mathscr{B}^0(S) \equiv \{ f \in \mathscr{B}_b(S), \langle f, \mu \rangle = 0\}$ is an invertible operator from $\mathscr{B}^0(S)$ into $\mathscr{B}^0(S)$. We denote its inverse by $(I - \pi)^{-1}$.

Lemma 4.1. *There exists a unique continuous extension* $\bar{\pi}$ *to* $L_0^2(\mu) \equiv \{ f \in L^2(\mu), \langle f, \mu \rangle = 0\}$ *of* π *on* $\mathscr{B}^0(S)$ *such that* $I - \bar{\pi}$ *is an invertible, bounded linear operator from* $L_0^2(\mu)$ *into* $L_0^2(\mu)$.

Proof. Since $\| \pi f \|_{L^2(\mu)} \leqslant \| f \|_{L^2(\mu)}$ and $\| (I - \pi) f \|_{L^2(\mu)} \leqslant 4 \| f \|_{L^2(\mu)}$ for all $f \in \mathscr{B}^0(S)$, it is easy to show that there is a unique operator $\bar{\pi}$ on $L_0^2(\mu)$ such that $\bar{\pi} f = \pi f$ for $f \in \mathscr{B}^0(S)$.

First, let us prove that $I - \bar{\pi}$ is one to one from $L_0^2(\mu)$ into $L_0^2(\mu)$. For $f \in L_0^2(\mu)$ satisfying $(I - \bar{\pi}) f = 0$, set $Z_n = f(X_n)$. Then $EZ_n^2 = \| f \|_{L^2(\mu)}^2 < +\infty$ and $E^{P_\mu}(Z_n | \mathscr{F}_{n-1}) = Z_{n-1}$ where $P_\mu = \int P_x \mu(\mathrm{d}x)$. Hence, $\{Z_n, \mathscr{F}_n, P_\mu\}$ is a square integrable martingale, and so $Z \equiv \lim_{n \to +\infty} Z_n$ exists. By positively recurrence, for $a < b$ satisfying $\mu(f < a) > 0$ and $\mu(f > b) > 0$, we have that for all $x \in S$.

$$P_x(X_n \in \{ f < a\} \ i.o.) = P_x(X_n \in \{ f > b\} \ i.o.) = 1 .$$

Hence, $f =$ constant. This and $\langle f, \mu \rangle = 0$ imply $f = 0$.

Next, let us prove that $I - \bar{\pi}$ is onto from $L_0^2(\mu)$ into $L_0^2(\mu)$. If π is aperiodic, there exists a constant $r > 0$ such that

$$\| \bar{\pi}^k f \|_{L^2(\mu)} \leqslant \mathrm{e}^{-rk} \| f \|_{L^2(\mu)}, \quad \text{for all } f \in L_0^2(\mu) \text{ and } k \geqslant 1 .$$

For each $f \in L_0^2(\mu)$, set

$$Rf \equiv f + \sum_{k=1}^{+\infty} \bar{\pi}^k f.$$

Then $Rf \in L_0^2(\mu)$ and $(I - \bar{\pi}) Rf = f$. Hence, in this case, $(I - \bar{\pi})$ is onto from $L_0^2(\mu)$ into $L_0^2(\mu)$.

If π is periodic, cU_c is Harris recurrent and satisfies Doeblin condition and μ is still the invariant probability measure of cU_c (cf. Revuz, 1984), where

$$U_c = \sum_{k=1}^{+\infty} (1 - c)^{k-1} \pi^k, \quad 0 < c < 1 .$$

Hence, $(I - \overline{cU_c})(L_0^2(\mu)) = (L_0^2(\mu))$. But on the other hand, the resolvent equation shows that

$$I - cU_c = (I - \pi)(I + (1 - c) U_c),$$

and so $(I - \bar{\pi})(L_0^2(\mu)) \supset (I - \overline{cU_c})(L_0^2(\mu)) = (L_0^2(\mu))$ which implies that $I - \bar{\pi}$ is onto.

Finally, since

$$\| (I - \bar{\pi}) f \|_{L^2(\mu)}^2 \leqslant 4 \| f \|_{L^2(\mu)}^2, \quad f \in L_0^2(\mu).$$

by Banach's theorem, $(I - \bar{\pi})$ is an invertible, bounded linear operator from $L_0^2(\mu)$ into $L_0^2(\mu)$ which completes the proof. □

Define

$$Rf = (I - \pi)^{-1}(f - \langle f, \mu \rangle), \quad f \in \mathscr{B}_b(S),$$

$$\Lambda(f) = \frac{1}{2}\langle (Rf)^2 - (\pi Rf)^2, \mu \rangle, \quad f \in \mathscr{B}_b(S),$$

and

$$\Lambda^*(\nu) = \sup_{y \in \mathscr{B}_b(S)} (\langle f, \nu \rangle - \Lambda(f)), \quad \nu \in \mathscr{B}_b^*(S).$$

Lemma 4.2. *If $\Lambda^*(\nu) < +\infty$, then $\nu \in M(S)$.*

Proof. It is enough to prove that for any $\{f_n\} \subset \mathscr{B}_b(S)$ satisfying $\sup_n \| f_n \| < +\infty$ and $\lim_{n \to +\infty} f_n(x) = 0$ for all $x \in S$,

$$\lim_{n \to +\infty} \Lambda(f_n) = 0.$$

In fact, since $I - \bar{\pi}$ is an invertible operator on $L_0^2(\mu)$, there exists a constant $C > 0$ such that

$$\| Rf \|_{L^2(\mu)} \leqslant C \| f - \langle f, \mu \rangle \|_{L^2(\mu)} \quad \text{for all } f \in \mathscr{B}_b(S).$$

Hence,

$$\limsup_{n \to +\infty} |\Lambda(f_n)| \leqslant \limsup_{n \to +\infty} \| Rf \|_{L^2(\mu)}^2 = 0.$$

Lemma is proved. □

Lemma 4.3. *For each $f \in \mathscr{B}_b(S)$, let*

$$M_n^f = Rf(X_n) - \pi Rf(X_{n-1}), \quad n \geqslant 1.$$

Then, for any $x \in S$, $\{M_n^f, \mathscr{F}_n, P_x\}$ is a bounded martingale difference sequence; and

$$\lim_{n \to +\infty} \sup_{x \in S} \sup_{j \geqslant 0} \left| \frac{1}{n} E^{P_x}\left(\sum_{i=1}^{n} (M_{j+i}^f)^2 \right) - 2\Lambda(f) \right| = 0.$$

Proof. By the Markov property, we can see easily that $\{M_n^f, \mathscr{F}_n, P_x\}$ is a martingale difference sequence. On the other hand,

$$\begin{aligned}\frac{1}{n} E^{P_x}\left(\sum_{i=1}^{n} (M_{j+i}^f)^2 \right) &= \frac{1}{n} \sum_{i=1}^{n} (\pi^{j+i}(Rf)^2(x) - \pi^{j+i-1}(\pi Rf)^2(x) \\ &= \frac{1}{n} \sum_{i=1}^{n} \pi^i(\pi^j(Rf)^2(x) - \frac{1}{n} \sum_{i=0}^{n-1} \pi^i(\pi^j(\pi Rf)^2(x).\end{aligned}$$

By (a), we obtain this lemma. □

Proof of Theorem 1.3. It is enough to prove (16). For any $f \in \mathscr{B}_b(S)$, by Lemma 4.3, Theorem 1.1 and the Markov property, we have that

$$\lim_{n\to+\infty} \sup_{x\in S} \left| \frac{n}{a^2(n)} \log E^{P_x}\left(\exp\left(\frac{a(n)}{n} \sum_{i=1}^{n} M_i^f\right)\right) - \Lambda(f) \right| = 0.$$

On the other hand,

$$\sum_{i=1}^{n} (f(X_i) - \langle f, \mu \rangle) = \sum_{i=1}^{n} M_i^f + Rf(X_0) - Rf(X_n).$$

Hence,

$$\limsup_{n\to+\infty} \sup_{x\in S} \left| \frac{n}{a^2(n)} \log E^{P_x}\left(\exp\left(\frac{a(n)}{n} \sum_{i=1}^{n} (f(X_i) - \langle f, \mu \rangle)\right)\right) - \Lambda(f) \right|$$
$$\leqslant \limsup_{n\to+\infty} \frac{2\|Rf\|}{a(n)} = 0.$$

which completes the proof of (16). □

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